On the sign of kurtosis near the QCD critical point

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We point out that the quartic cumulant (and kurtosis) of the order parameter fluctuations is universally *negative* when the critical point is approached on the crossover side of the phase separation line. As a consequence, the kurtosis of a fluctuating observable, such as, e.g., proton multiplicity, may become smaller than the value given by independent Poisson statistics. We discuss implications for the Beam Energy Scan program at RHIC.

INTRODUCTION

Mapping the QCD phase diagram as a function of temperature T and baryochemical potential μ_B is one of the fundamental goals of heavy-ion collision experiments. QCD critical point is a distinct singular feature of the phase diagram. It is a ubiquitous property of QCD models based on the chiral symmetry breaking dynamics (see, e.g., Ref.[1] for a review and further references). Locating the point using first-principle lattice calculations is a formidable challenge (see, e.g., Ref.[2] for a recent review and references). If the critical point is situated in the region accessible to heavy-ion collision experiments it can be discovered experimentally. The search for the critical point is planned at the Relativistic Heavy Ion Collider (RHIC) at BNL, the Super Proton Synchrotron (SPS) at CERN, the future Facility for Antiproton and Ion Research (FAIR) at GSI, and Nuclotron-based Ion Collider Facility (NICA) in Dubna (see, e.g., Ref.[3]).

The characteristic feature of a critical point is the divergence of the correlation length ξ and of the magnitude of the fluctuations. The simplest measures of fluctuations in heavy-ion collisions are the variances of the event-by-event observables such as multiplicities or mean transverse momenta of particles. The singular, critical contribution to these variances diverges as (approximately) ξ^2, and would manifest in a non-monotonic dependence of such measures as the critical point is passed by during the beam energy scan [4, 5]. In realistic heavy ion collision the divergence of ξ is cut-off by the effects of critical slowing down [5, 6], and the estimates of the maximum correlation length are in the range of at most $2-3$ fm, compared to the natural $0.5-1$ fm away from the critical point. However, higher, non-Gaussian, moments of the fluctuations depend much more sensitively on ξ, according to Ref.[7]. For example, the 4-th moment grows as ξ^7 near the critical point, making it an attractive experimental tool. In this paper we follow up on the results of Ref.[7] to point out that the sign of the 4-th moment could be negative as the critical point is approached from the crossover side of the QCD phase transition.

The sign of various moments have been discussed in the literature in related contexts: see, e.g., discussion of the sign of the 3-rd moment in Ref.[8] or the 6-th and 8-th moments in Ref.[9] and also numerical lattice calculations in Ref.[10] where the possible sign change of kurtosis is noted.

In this paper we shall address specifically the sign of the 4-th moment (or kurtosis) and do it in a more universal and quantitative way than has been done previously, by using the known parametric form of the universal equation of state near the critical point. We emphasize universality of the behavior of the kurtosis and draw experimental consequences from these results.

KURTOSIS AND UNIVERSAL EFFECTIVE POTENTIAL

Let us begin, as in Ref.[1], by describing fluctuations of the order parameter field $\sigma(\boldsymbol{x})$ near a critical point using the probability distribution

$$P[\sigma] \sim \exp\left\{-\Omega[\sigma]/T\right\}, \tag{1}$$

where Ω is the effective action (free energy) functional for the field σ, which can be expanded in powers of σ as well as in the gradients (we chose $\sigma = 0$ at the minimum):

$$\Omega = \int d^3\boldsymbol{x} \left[\frac{(\boldsymbol{\nabla}\sigma)^2}{2} + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \dots\right]. \tag{2}$$

Calculating 2-point correlator $\langle\sigma(\boldsymbol{x})\sigma(0)\rangle$ we find that the correlation length $\xi = m_\sigma^{-1}$. For the moments of the zero momentum mode $\sigma_V \equiv \int d^3x\, \sigma(x)$ in a system of volume V we find at tree level

$$\begin{aligned} &\kappa_2 = \langle\sigma_V^2\rangle = VT\,\xi^2\,; \qquad \kappa_3 = \langle\sigma_V^3\rangle = 2\lambda_3 VT^2\,\xi^6\,; \\ &\kappa_4 = \langle\sigma_V^4\rangle_c = 6VT^3\left[\,2(\lambda_3\xi)^2 - \lambda_4\,\right]\xi^8\,. \end{aligned} \tag{3}$$

where $\langle\sigma_V^4\rangle_c \equiv \langle\sigma_V^4\rangle - 3\langle\sigma_V^2\rangle^2$ denotes the connected 4-th central moment (the 4-th cumulant). The critical point is characterized by $\xi \to \infty$. The central observation in Ref.[7] was that the higher moments (cumulants) κ_3 and κ_4 diverge with ξ much faster than the quadratic moment κ_2. Here we shall point out that the *sign* of the 4-th moment κ_4 is negative in a certain sector near the critical point. More precisely, the 4-th cumulant is negative when the critical point is approached from the crossover side. Let us demonstrate this in several complementary ways.

A simple way to see why the kurtosis is negative is by following the evolution of the probability distribution of

σ_V as we approach the critical point along the crossover line. In Ising scaling coordinates: along $H = 0$, $t > 0$ ray. Away from the critical point, more precisely for $\xi^3 \ll V$, the central limit theorem dictates that the probability distribution of σ_V is Gaussian, with a vanishingly small kurtosis. As we approach the critical point the distribution develops non-Gaussian shape. This intermediate shape is a deformation of the Gaussian towards a two-peak distribution, corresponding to the phase coexistence on the opposite, first-order transition side ($t < 0$) of the critical point. Such a shape is clearly less "peaked" than the Gaussian, and thus corresponds to negative kurtosis.

More quantitatively, the kurtosis vanishes as $1/V$ at (almost) any point away from the critical point, i.e.,

$$K \equiv \kappa_4/\kappa_2^2 = \mathcal{O}(\xi^3/V). \tag{4}$$

The exception is the coexistence line ($H = 0$, $t < 0$ ray). The distribution there has two peaks of equal height and its kurtosis is $K = -2+\mathcal{O}(\xi^3/V)$.[1] It is important to note that this is only true strictly *on* the coexistence line $H = 0$, for the moments measured around the symmetric point of the probability distribution of σ_V, which is actually a dip, not a peak, for $t < 0$. At any point close to the coexistence line, i.e., at $H \neq 0$, $t < 0$, the kurtosis around the dominant peak is positive.

In the scaling regime (close to, but not at the critical point) where ξ is much greater than the microscopic scale, a, but still much less than the linear size of the system: $a \ll \xi \ll V^{1/3}$, the coefficient of ξ^3/V in Eq. (4) can be expressed in terms of the couplings λ_i using Eqs. (3):

$$K = 6\left(2\lambda_3^2\xi^3 - \lambda_4\xi\right)\frac{\xi^3}{V}. \tag{5}$$

These couplings, and in fact the shape of the effective potential, is also universal. In particular, λ_4 scales with ξ as $\lambda_4 = \tilde{\lambda}_4\xi^{-1}$, where the universal value of $\tilde{\lambda}_4$ is known approximately to be 4.0 on the crossover line (see, e.g., Ref.[11] for a review).[2] Since on the crossover line $\lambda_3 = 0$ and $\lambda_4 > 0$, it is clear from Eq. (3) that $K < 0$.

Away from the crossover line ($H = 0$, $t > 0$ ray) the distribution is skewed: $\lambda_3 \neq 0$. This makes the kurtosis positive, according to Eq. (5), except for a certain sector around the crossover line.

[1] In fact, this transition of the shape of the distribution around a critical point is universal to all critical points of the same (Ising model) universality class. At finite V, at $t = 0$, the value of K is independent of V (the correlation length $\xi \sim V^{1/3}$ is as large as it can be at given volume). This value of K is a universal number (it depends only on the boundary conditions for a given universality class) and is well-known. It is usually expressed as the value of the Binder cumulant $B_4 \approx 1.6$, which means $K = B_4 - 3 \approx -1.4 < 0$.

[2] As in Ref.[7], for simplicity and consistency with our overall level of precision, we neglect the anomalous scaling dimension η, which is only of order few percent.

THE UNIVERSAL EQUATION OF STATE AROUND THE CRITICAL POINT

To extend this analysis away from the crossover line, i.e., to take into account $\lambda_3 \neq 0$ in Eq. (5), we need to know the equation of state, in particular, κ_4 as a function of both Ising variables: reduced temperature t and magnetic field H. In the scaling regime near $t = H = 0$ this equation of state is also universal. For the Ising model it is known to order ε^3 in the epsilon expansion as well as numerically.

Before we discuss this universal form, let us keep in mind that the mapping of QCD phase diagram in the T, μ_B plane into t, H plane is not universal. However, this mapping is analytic, i.e., both functions $t(T, \mu_B)$ and $H(T, \mu_B)$ are analytic at the critical point, which is mapped into the origin, $t(T^{\rm cp}, \mu_B^{\rm cp}) = H(T^{\rm cp}, \mu_B^{\rm cp}) = 0$.

The standard parametrization, Ref.[13], of the equation of state in the scaling domain near the critical point is in terms of two new scaling variables R and θ (it has been applied in the context of QCD before, Ref.[14]). Denoting the "magnetization" by $M = \langle\sigma_V\rangle/V$ we define R and θ as

$$M = R^\beta\theta, \quad t = R(1-\theta^2), \tag{6}$$

Then the equation of state can be expressed in terms of the single function $h(\theta)$ as

$$H = R^{\beta\delta}h(\theta). \tag{7}$$

Unlike the explicit form of the singular equation of state $M = M(t, H)$, the function $h(\theta)$ is analytic. It has two zeros. One, at $\theta = 0$, corresponds to the crossover line ($t > 0$, $H = 0$ ray), another, at some $\theta = \theta_1 > 1$, corresponds to the coexistence (first-order transition) line ($t < 0$, $H = 0$ ray). The function $h(\theta)$ must also be odd since $M(-H) = -M(H)$. The simplest function obeying all these requirements is a cubic polynomial

$$h(\theta) = \theta(2 - 3\theta^2). \tag{8}$$

where the value $\theta_1 = \sqrt{3/2}$ is a good approximation to the universal value for the Ising model (and correct up to $\mathcal{O}(\varepsilon^2)$). The choice (8) is known as the linear parametric model, Ref.[12]. It describes the equation of state with precision quite sufficient for our purposes. The linear parametric model is also known to be exact up to $\mathcal{O}(\varepsilon^3)$.

Using this parametric equation of state, we can calculate the moments by taking derivatives at fixed t, up to an overall normalization, unimportant in the present context (it can be fixed by Eq. (3)). In particular,

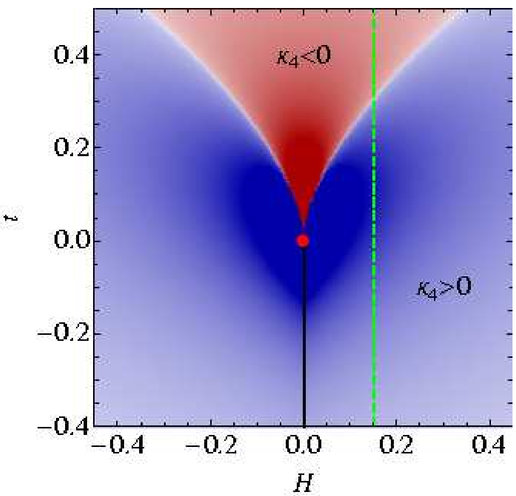
$$\kappa_4(t, H) = \left(\frac{\partial^3 M}{\partial H^3}\right)_t. \tag{9}$$

For our purposes, it would be sufficient to use the approximate rational values of critical exponents $\beta = 1/3$

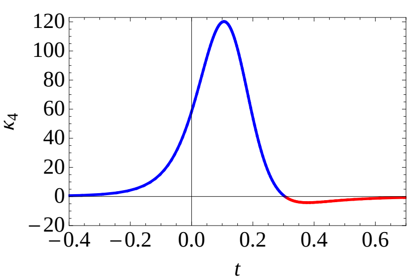
and $\delta = 5$, which are within few percent of their exact values in three dimensions. The result of Eq. (9) can then be simplified to

$$\kappa_4(t,H) = -12\,\frac{81 - 783\theta^2 + 105\theta^4 - 5\theta^6 + 2\theta^8}{R^{14/3}(3-\theta^2)^3(3+2\theta^2)^5}\,. \quad (10)$$

We represent $\kappa_4(t,H)$ graphically as a density plot in Fig. 1. We see that the 4-th cumulant (and kurtosis) is negative in the sector bounded by two curved rays $H/t^{\beta\delta} = \pm\text{const}$ (corresponding to $\theta \approx \pm 0.32$).



(a)



(b)

FIG. 1: (color online) (a) – the density plot of the function $\kappa_4(t,H)$ given by Eq. (10) obtained using Eq. (9) for the linear parametric model Eqs. (6), (7), (8) and $\beta = 1/3$, $\delta = 5$. The $\kappa_4 < 0$ region is red, the $\kappa_4 > 0$ – is blue. (b) – the dependence of κ_4 on t along the vertical dashed green line on the density plot above. This line is the simplest example of a possible mapping of the freezeout curve (see Fig. 2). The units of t, H and κ_4 are arbitrary.

Also in Fig. 1 we show the dependence of κ_4 along a line which could be thought of as representing a possible mapping of the freezeout trajectory (Fig. 2) onto the tH plane. Although the absolute value of the peak in κ_4 depends on the proximity of the freezeout curve to the critical point, the ratio of the maximum to minimum along such an $H = \text{const}$ curve is a universal number, approximately equal to -28 from Eq. (10).

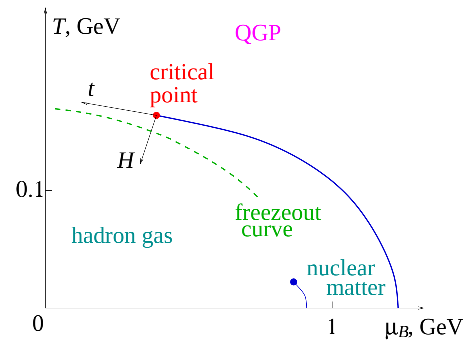


FIG. 2: A sketch of the phase diagram of QCD with the freezeout curve and a possible mapping of the Ising coordinates t and H.

The negative minimum is small relative to the positive peak, but given the large size of the latter, Ref.[7, 15], the negative contribution to kurtosis may be significant. In addition, the mapping of the freezeout curve certainly need not be $H = \text{const}$, and the relative size of the positive and negative peaks depends sensitively on that.

The trend described above appears to show in the recent lattice data, Ref.[10], obtained using Pade resummation of the truncated Taylor expansion in μ_B. As the chemical potential is increased along the freezeout curve, the 4-th moment of the baryon number fluctuations begins to decrease, possibly turning negative, as the critical point is approached (see Fig.2 in Ref.[10]).

Another observation, which we shall return to at the end of the next section, is that $-\kappa_4$ grows as we approach the crossover line, corresponding to $H = 0$, $t > 0$ on the diagram in Fig. 1(a). On the QCD phase diagram the freezeout point will move in this direction if one reduces the size of the colliding nuclei or selects more peripheral collisions (the freezeout occurs earlier, i.e., at higher T, in a smaller system).

EXPERIMENTAL OBSERVABLES

In this section we wish to connect the results for the fluctuations of the order parameter field σ to the fluctuations of the observable quantities. As an example we consider the fluctuations of the multiplicity of given charged particles, such as pions or protons.

For completeness we shall briefly rederive the results of Ref.[7] using a simple model of fluctuations. The model captures the most singular term in the contribution of the critical point to the fluctuation observables. Consider a given species of particle interacting with fluctuating critical mode field σ. The infinitesimal change of the field $\delta\sigma$ leads to a change of the effective mass of the particle by the amount $\delta m = g\delta\sigma$. This could be considered a definition of the coupling g. For example, the coupling of protons in the sigma model is $g\sigma\bar{p}p$. The fluctuations $\delta f_{\boldsymbol{p}}$

of the momentum space distribution function $f_{\boldsymbol{p}}$ consist of the pure statistical fluctuations $\delta f^0_{\boldsymbol{p}}$ around the equilibrium distribution $n_{\boldsymbol{p}}$ for a particle of a given mass, which itself fluctuates. This gives

$$\delta f_{\boldsymbol{p}} = \delta f^0_{\boldsymbol{p}} + \frac{\partial n_{\boldsymbol{p}}}{\partial m}\, g\, \delta\sigma\,. \tag{11}$$

Using this equation we can calculate the most singular contribution from the critical fluctuations to the moments or correlators of $\delta f_{\boldsymbol{p}}$. The fluctuation of the multiplicity $N = V\, d \int_{\boldsymbol{p}} f_{\boldsymbol{p}}$ is given by

$$\delta N = \delta N^0 + V\, g\, \delta\sigma\, d \int_{\boldsymbol{p}} \frac{\partial n_{\boldsymbol{p}}}{\partial m}\,, \tag{12}$$

where d is the degeneracy factor (e.g., number of spin or charge states of the particle). Neglecting, for clarity and simplicity, the effects of quantum statistics, i.e., assuming $n_{\boldsymbol{p}} \ll 1$, we can use Poisson statistics for δN^0. Using additivity of the cumulants (their defining property), and assuming δN^0 and $\delta\sigma$ are uncorrelated, the contribution of the critical fluctuations can be expressed in terms of the corresponding moments of the critical field σ fluctuations. For example, the contribution to the 4-th moment can be expressed as (cf. Refs.[7, 15])

$$\langle(\delta N)^4\rangle_c = \langle N\rangle + \langle\sigma_V^4\rangle_c \left(\frac{g\, d}{T} \int_{\boldsymbol{p}} \frac{n_{\boldsymbol{p}}}{\gamma_{\boldsymbol{p}}}\right)^4 + \ldots, \tag{13}$$

where $\gamma_{\boldsymbol{p}} = (dE_{\boldsymbol{p}}/dm)^{-1}$ is the relativistic gamma-factor of a particle with momentum $\boldsymbol{p}$ and mass m. The first term on the r.h.s. of Eq. (13) is the Poisson contribution. We neglected $n_{\boldsymbol{p}} \ll 1$ in the quantum statistics factor $(1 \pm n_{\boldsymbol{p}})$ for simplicity, and we denoted by "…" other contributions, less singular at the critical point. The model is admittedly crude, but it illustrates the mechanism and correctly captures the most singular contribution near the critical point.

In the region near the critical point where $\kappa_4 = \langle\sigma_V^4\rangle_c$ is negative, the 4-th cumulant of the fluctuations will be smaller than its Poisson value, $\langle N\rangle$. The measure defined in Ref.[7] as $\omega_4(N) = \langle(\delta N)^4\rangle_c/\langle N\rangle$ will be less than 1. By how much will depend sensitively on the correlation length (as ξ^7), i.e., on how close the freezeout occurs to the critical point, as well as on other factors (for protons, most significantly, on the value of μ_B.) We shall not attempt to estimate this effect quantitatively in this paper. The analysis of Ref.[15] suggests, however, that this effect for protons can be significant compared to the Poisson value already for $\xi \sim 2$ fm.

Usual caveats apply: other (non-trivial) contributions to moments which do not behave singularly at the critical point can turn out to be relatively large. It is beyond the scope of the paper to estimate these effects. The size of these background contributions could, in principle, be determined experimentally by performing measurements away from the critical point.

We conclude by asking an obvious question: has the effect of the negative kurtosis been observed? Data from STAR indicate that at $\sqrt{s} = 19.6$ GeV the ratio κ_4/κ_2 might be substantially smaller than its Poisson value 1, see Fig. 6 in Ref.[16], while it is very close to 1 at higher $\sqrt{s}$ (smaller μ_B). Unfortunately, the statistics accumulated in the short run at $\sqrt{s} = 19.6$ GeV is clearly not sufficient to make a reliable conclusion. It would be interesting to see if this effect persists with more statistics at this energy. If confirmed, this result could indicate that the critical point is close, at somewhat larger values of μ_B (smaller $\sqrt{s}$). In this case, as we already discussed at the end of the previous section, the universality would also predict that the negative kurtosis effect should increase in more peripheral collisions at the same $\sqrt{s}$. At smaller values of $\sqrt{s}$ the effect should change sign, increasing kurtosis above its Poisson value.

Acknowledgments

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